Fundamentals of Actuarial Mathematics

Third Edition

S. David Promislow

$$\frac{d}{dt}\,{}_t\bar{V} = \delta({}_t\bar{V}) + \pi(t) - \mu_x(t)\left(b(t) - {}_t\bar{V}\right)$$

WILEY

Fundamentals of Actuarial Mathematics

Fundamentals of Actuarial Mathematics

Third Edition

S. David Promislow

York University, Toronto, Canada

WILEY

This edition first published 2015

Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

For details of our global editorial offices, for customer services and for information about how to apply for permission to reuse the copyright material in this book please see our website at www.wiley.com.

Library of Congress Cataloging-in-Publication Data

Promislow, S. David.
Fundamentals of actuarial mathematics / S. David Promislow. – Third edition.
pages cm
Includes bibliographical references and index.
ISBN 978-1-118-78246-0 (hardback)
1. Insurance–Mathematics. 2. Business mathematics. I. Title.
HG8781.P76 2014
368′.01–dc23
2014027082

A catalogue record for this book is available from the British Library.

ISBN: 9781118782460

Set in 10/12pt Times by Aptara Inc., New Delhi, India

1 2015

To Georgia and Griffith

Contents

Preface

The third edition of this book continues the objective of providing coverage of actuarial mathematics in a flexible manner that meets the needs of several audiences. These range from those who want only a basic knowledge of the subject, to those preparing for careers as professional actuaries. All this is carried out with a streamlined system of notation, and a modern approach to computation involving spreadsheets.

The text is divided into four parts. The first two cover the subject of life contingencies. The modern approach towards this subject is through a stochastic model, as opposed to the older deterministic viewpoint. I certainly agree that mastering the stochastic model is the desirable goal. However, my classroom experience has convinced me that this is not the right place to begin the instruction. I find that students are much better able to learn the new ideas, the new notation, the new ways of thinking involved in this subject, when done first in the simplest possible setting, namely a deterministic discrete model. After the main ideas are presented in this fashion, continuous models are introduced. In Part II of the book, the full stochastic model of life contingencies can be dealt with in a reasonably quick fashion.

Another innovation in Part II is to depart from the conventional treatment of life contingencies as dealing essentially with patterns of mortality or disability in a group of human lives. Throughout Part II, we deal with general *failure times* which makes the theory more widely adaptable.

Part III deals with more advanced stochastic models. Following an introduction to stochastic processes, there is a chapter covering multi-state theory, an approach which unifies many of the ideas in Parts I and II. The final chapter in Part III is an introduction to modern financial mathematics.

Part IV deals with the subject of risk theory, sometime referred to as loss models. It includes an extensive coverage of classical ruin theory, a topic that originated in actuarial science but recently has found many applications in financial economics. It also includes credibility theory, which will appeal to the reader interested more in the casualty side of actuarial mathematics.

This book will meet the needs of those preparing for the examinations of many of the major professional actuarial organizations. Parts I to III of this new third edition covers all of the material on the current syllabuses of Exam MLC of the Society of Actuaries and Canadian Institute of Actuaries and Exam LC of the Casualty Actuarial Society, and covers most of the topics on the current syllabus of Exam CT5 of the British Institute of Actuaries.

In addition, Part IV of the book covers a great deal of the material on Exam C of the Society of Actuaries and Canadian Institute of Actuaries, including the topics of Frequency, Severity and Aggregate Models, Risk Measures, and Credibility Theory.

The mathematical prerequisites for Part 1 are relatively modest. comprising elementary linear algebra and probability theory, and, beginning in Chapter 8, some basic calculus. A more advanced knowledge of probability theory is needed from Chapter 13 onward, and this material summarized in Appendix A. A usual prerequisite for actuarial mathematics is a course in the theory of interest. Although this may be useful, it is not strictly required. All the interest theory that is needed is presented as a particular case of the general deterministic actuarial model in Chapter 2.

A major source of difficulty for many students in learning actuarial mathematics is to master the rather complex system of actuarial notation. We have introduced some notational innovations, which tie in well with modern calculation procedures as well as allow us to greatly simplify the notation that is required. We have, however, included all the standard notation in separate sections, at the end of the relevant chapters, which can be read by those readers who desire this material.

Keeping in mind the nature of the book and its intended audience, we have avoided excessive mathematical rigour. Nonetheless, careful proofs are given in all cases where these are thought to be accessible to the typical senior undergraduate mathematics student. For the few proofs not given in their entirety, mainly those involving continuous-time stochastic processes, we have tried at least to provide some motivation and intuitive reasoning for the results.

Exercises appear at the end of each chapter. In Parts I and II these are divided up into different types. Type A exercises generally are those which involve direct calculation from the formulas in the book. Type B involve problems where more thought is involved. Derivations and problems which involve symbols rather than numeric calculation are normally included in Type B problems. A third type is spreadsheet exercises which themselves are divided into two subtypes. The first of these asks the reader to solve problems using a spreadsheet. Detailed descriptions of applicable Microsoft Excel® spreadsheets are given at the end of the relevant chapters. Readers of course are free to modify these or construct their own. The second subtype does not ask specific questions but instead asks the reader to modify the given spreadsheets to handle additional tasks. Answers to most of the calculation-type exercises appear at the end of the book.

Sections marked with an asterisk * deal with more advanced material, or with special topics that are not used elsewhere in the book. They can be omitted on first reading. The exercises dealing with such sections are likewise marked with *, as are a few other exercises which are of above average difficulty.

There are various ways of using the text for university courses geared to third or fourth year undergraduates, or beginning graduate students. Chapters 1 to 8 could form the basis of a one-semester introductory course. Part IV is for the most part independent of the first three parts, except for the background material on stochastic processes given in Chapter 18 and would constitute another one-semester course. The rest of the book constitutes roughly another two semesters worth of material, with possibly some omissions; Chapter 13 is not needed for the rest of the book. Chapters 7 (except for Section 7.3.1), 9 and 12 deal with topics that are important in applications, but which are used minimally in other parts of the text. They could be omitted without loss of continuity.

CHANGES IN THE THIRD EDITION

There are several additions and changes to the third edition.

The most notable is a new Chapter 20 providing an introduction to the mathematics of financial markets. It has been long recognized that knowledge of this subject is essential to the management of financial risk that faces the actuary of today.

Other additions include the following:

- Chapter 12, on expenses, has been considerably enlarged to include the topic of profit testing.
- The chapter on multi-state models has been expanded to include discussion of reserves and profit testing in such models, as well as several additional techniques for continuous-time problems.
- Some extra numerical procedures have been included, such as Euler's method for differential equations, and the three-term Woolhouse formulas for fractional annuity approximations.
- An introduction to Brownian motion has been added to the material on continuous-time stochastic processes.
- The previous material on universal life and variable annuities has been rewritten and included in a new chapter dealing with miscellaneous topics. A brief discussion of pension plans is included here as well.
- Additional examples, exercises, and clarification have been added to various chapters.

As well as the changes there has been a reorganization in the material The previous two chapters on stochastic processes have been combined into one and now appear earlier in the book as background for the multi-state and financial markets chapters. In the current Part IV, the detailed descriptions of the various distributions have been removed and added as a section to the Appendix on probability theory.

Acknowledgements

Several individuals have assisted in the various editions of this book. I am particularly indebted to two people who have made a significant contribution by providing a number of helpful comments, corrections, and suggestions. They are Virginia Young for her work on the first edition, and Elias Shiu for his help with the third edition.

There are many others who deserve thanks. Moshe Milevsky provided enlightening comments on annuities and it was his ideas that motivated the credit risk applications in Chapter 10, as well as some of the material on generational annuity tables in Chapter 9. Several people found misprints in the first edition and earlier drafts. These include Valerie Michkine, Jacques Labelle, Karen Antonio, Kristen Moore, as well as students at York University and the University of Michigan. Christian Hess asked some questions which led to the inclusion of Example 21.10 to clear up an ambiguous point. Exercise 18.13 was motivated by Bob Jewett's progressive practice routines for pool. My son Michael, a life insurance actuary, provided valuable advice on several practical aspects of the material. Thanks go to the editorial and production teams at Wiley for their much appreciated assistance. Finally, I thank my wife Shirley who provided support and encouragement throughout the writing of all three editions.

About the companion website

This book is accompanied by a companion website:

www.wiley.com/go/promislow/actuarial

The website includes:

- A variety of exercises, both computational and theoretical
- Answers, enabling use for self-study.

Part I

THE DETERMINISTIC LIFE CONTINGENCIES MODEL

1

Introduction and motivation

1.1 Risk and insurance

In this book we deal with certain mathematical models. This opening chapter, however, is a nontechnical introduction, designed to provide background and motivation. In particular, we are concerned with models used by actuaries, so we might first try to describe exactly what it is that actuaries do. This can be difficult, because a typical actuary is concerned with many issues, but we can identify two major themes dealt with by this profession.

The first is *risk*, a word that itself can be defined in different ways. A commonly accepted definition in our context is that risk is the possibility that *something bad* happens. Of course, many bad things can happen, but in particular we are interested in occurrences that result in *financial loss*. A person dies, depriving family of earned income or business partners of expertise. Someone becomes ill, necessitating large medical expenses. A home is destroyed by fire or an automobile is damaged in an accident. No matter what precautions you take, you cannot rid yourself completely of the possibility of such unfortunate events, but what you can do is take steps to mitigate the financial loss involved. One of the most commonly used measures is to purchase insurance.

Insurance involves a sharing or pooling of risks among a large group of people. The origins go back many years and can be traced to members of a community helping out others who suffered loss in some form or other. For example, people would help out neighbours who had suffered a death or illness in the family. While such aid was in many cases no doubt due to altruistic feelings, there was also a motivation of self-interest. You should be prepared to help out a neighbour who suffered some calamity, since you or your family could similarly be aided by others when you required such assistance. This eventually became more formalized, giving rise to the insurance companies we know today.

With the institution of insurance companies, sharing is no longer confined to the scope of neighbours or community members one knows, but it could be among all those who chose to purchase insurance from a particular company. Although there are many different types

Fundamentals of Actuarial Mathematics, Third Edition. S. David Promislow.

Companion website: http://www.wiley.com/go/actuarial

of insurance, the basic principle is similar. A company known as the *insurer* agrees to pay out money, which we will refer to as *benefits*, at specified times, upon the occurrence of specified events causing financial loss. In return, the person purchasing insurance, known as the *insured*, agrees to make payments of prescribed amounts to the company. These payments are typically known as *premiums*. The contract between the insurer and the insured is often referred to as the *insurance policy*.

The risk is thereby transferred from the individuals facing the loss to the insurer. The insurer in turn reduces its risk by insuring a sufficiently large number of individuals, so that the losses can be accurately predicted. Consider the following example, which is admittedly vastly oversimplified but designed to illustrate the basic idea.

Suppose that a certain type of event is unlikely to occur but if so, causes a financial loss of 100 000. The insurer estimates that about 1 out of every 100 individuals who face the possibility of such loss will actually experience it. If it insures 1000 people, it can then expect 10 losses. Based on this model, the insurer would charge each person a premium of 1000. (We are ignoring certain factors such as expenses and profits.) It would collect a total of 1 000 000 and have precisely enough to cover the 100 000 loss for each of the 10 individuals who experience this. Each individual has eliminated his or her risk, and in so far as the estimate of 10 losses is correct, the insurer has likewise eliminated its own risk. (We comment further on this statement in the next section.)

We conclude this section with a few words on the connection between insurance and gambling. Many people believe that insurance is really a form of the latter, but in fact it is exactly the opposite. Gambling trades certainty for uncertainty. The amount of money you have in your pocket is there with certainty if you do not gamble, but it is subject to uncertainty if you decide to place a bet. On the other hand, insurance trades uncertainty for certainty. The uncertain drain on your wealth, due to the possibility of a financial loss, is converted to the certainty of the much smaller drain of the premium payments if you insure against the loss.

1.2 Deterministic versus stochastic models

The example in Section 1.1 illustrates what is known as a *deterministic* model. The insurer in effect pretends it will know exactly how much it will pay out in benefits and then charges premiums to match this amount. Of course, the insurer knows that it cannot really predict these amounts precisely. By selling a large number of policies they hope to benefit from the diversification effect. They are really relying on the statistical concept known as the 'law of large numbers', which in this context intuitively says that if a sufficiently large number of individuals are insured, then the total number of losses will likely be close to the predicted figure.

To look at this idea in more detail, it may help to give an analogy with flipping coins. If we flip 100 fair coins, we cannot predict exactly the number of them that will come up heads, but we expect that most of the time this number should be close to 50. But 'most of the time' does not mean always. It is possible for example, that we may get only 37 heads, or as many as 63, or even more extreme outcomes. In the example given in the last section, the number of losses may well turn out to be more than the expected number of 10. We would like to know just how unlikely these rare events are. In other words, we would like to quantify more precisely just what the words 'most of the time' mean. To achieve this greater sophistication a stochastic model for insurance claims is needed, which will assign probabilities to the occurrence of various numbers of losses. This will allow adjustment of premiums in order to allow for the risk that the actual number of losses will deviate from that expected. We will however begin the study of actuarial mathematics by first developing a deterministic approach, as this seems to be the best way of learning the basic concepts. After mastering this, it is not difficult to turn to the more realistic stochastic setting.

We will not get into all the complications that can arise. In actual coin flipping it seems clear that the results of each toss are independent of the others. The fact that one coin comes up heads, is not going to affect the outcomes of the others. It is this independence which is behind the law of large numbers, and which results in outcomes that are usually close to what is expected. There are some risks, often referred to as *systematic* or *non-diversifiable,* where the independence assumption fails, and which can adversely affect all or a large number of members of a group at the same time. For example, a spreading epidemic could cause life or health insurers to pay more in claims than they expected. Selling more policies in order to diversify would not help their financial situation. It could in fact make it worse, if the premiums were not sufficient to cover the extra losses. Severe climatic disturbances causing storms could impact property insurance in the same way. In 2008, falling real estate prices in the United States affected mortgage lenders and those who insured mortgage lenders against bad debts, to the extent that this helped trigger a global financial crisis. A detailed discussion of these matters is not within the scope of this work, and for the most part, the stochastic model we present will confine attention to the usual insurance model where the risks are considered as independent. It should be kept in mind however that the detection and avoidance of systematic risk are matters that the actuary must always be aware of.

1.3 Finance and investments

The second theme involved in an actuary's work is finance and investments. In most of the types of insurance that we focus on in this book, an additional complicating factor is the long-term nature of the contracts. Benefits may not be paid until several years after premiums are collected. This is certainly true in life insurance, where the loss is occasioned by the death of an individual. Premiums received are invested and the resulting earnings can be used to help provide the benefits. Consider the simple example given above, and suppose further that the benefits do not have to be paid until 1 year after the premiums are collected. If the insurer can invest the money at, say, 5% interest for the year, then it does not need to charge the full 1000 in premium, but can collect only 1000/1.05 from each person. When invested, this amount will provide the necessary 1000 to cover the losses. Again, this example is oversimplified and there are many more complications. We will, in the next chapter, consider a mathematical model that deals with the consequences of the payments of money at various times. A much more elaborate treatment of financial matters, incorporating randomness, is presented in Chapter 20.

1.4 Adequacy and equity

We can now give a general description of the responsibilities of an actuary. The overriding task is to ensure that the premiums, together with investment earnings, are *adequate* to provide for the payment of the benefits. If this is not true, then it will not be possible for the insurer to meet its obligations and some of the insureds will necessarily not receive compensation for their losses. The challenge in meeting this goal arises from the several areas of uncertainty. The amount and timing of the benefits that will have to be paid, as well as the investment earnings, are unknown and subject to random fluctuations. The actuary makes substantial use of probabilistic methods to handle this uncertainty.

Another goal is to achieve *equity* in setting premiums. If an insurer is to attract purchasers, it must charge rates that are perceived as being fair. Here also, the randomness means that it is not obvious how to define equity in this context. It cannot mean that two individuals who are charged the same amount in premiums will receive exactly the same back in benefits, for that would negate the sharing arrangement inherent in the insurance idea. While there are different possible viewpoints, equity in insurance is generally expected to mean that the mathematical expectation of these two individuals should be the same.

1.5 Reassessment

Actuaries design insurance contracts and must initially calculate premiums that will fulfill the goals of adequacy and equity, but this is not the end of the story. No matter how carefully one makes an initial assessment of risks, there are too many variables to be able to achieve complete accuracy. Such assessments must be continually re-evaluated, and herein lies the real expertise of the actuary. This work may be compared to sailing a ship in a stormy sea. It is impossible to avoid being blown off course occasionally. The skill is to detect when this occurs and to take the necessary steps to continue in the right direction. This continual monitoring and reassessing is an important part of the actuary's work. A large part of this involves calculating quantities known as *reserves*. We introduce this concept in Chapter 2 and then develop it more fully in Chapter 6.

1.6 Conclusion

We can now summarize the material found in the subsequent chapters of the book. We will describe the mathematical models used by the actuary to ensure that an insurer will be able to meet its promised benefits payments and that the respective purchasers of its contracts are treated equitably. In Part I, we deal with a strictly deterministic model. This enables us to focus on the main principles while keeping the required mathematics reasonably simple. In Part II, we look at the stochastic model for an individual insurance contract. In Part III, we look at more advanced stochastic models and introduce the mathematics of financial markets. In Part IV, we consider models that encompass an entire portfolio of insurance contracts.

2

The basic deterministic model

2.1 Cash flows

As indicated in the previous chapter, a basic application of actuarial mathematics is to model the transfer of money. Insurance companies, banks and other financial institutions engage in transactions that involve accepting sums of money at certain times, and paying out sums of money at other times.

To construct a model for describing this situation, we will first fix a time unit. This can be arbitrary, but in most applications it will be taken as some familiar interval of time. For convenience we will assume that time is measured in years, unless we indicate otherwise. We will let time 0 refer to the present time, and time t will then denote t time units in the future. We also select an arbitrary unit of capital. In this chapter, we assume that all funds are paid out or received at integer time points, that is, at time $0, 1, 2, \ldots$. The amount of money received or paid out at time k will be called the *net cash flow* at time k and denoted by c_k. A positive value of c_k denotes that a sum is to be received, whereas a negative value indicates that a sum is paid out. The entire transaction is then described by listing the sequence of cash flows. We will refer to this as a *cash flow vector*,

$$\mathbf{c} = (c_0, c_1, \ldots, c_N),$$

where N is the final duration for which a payment is made.

For example, suppose I lend you 10 units of capital now and a further 5 units a year from now. You repay the loan by making three yearly payments of 7 units each, beginning 3 years from now. The resulting cash flow vector from my point of view is

$$\mathbf{c} = (-10, -5, 0, 7, 7, 7).$$

From your point of view, the transaction is represented by $-\mathbf{c} = (10, 5, 0, -7, -7, -7)$.

Fundamentals of Actuarial Mathematics, Third Edition. S. David Promislow.

Companion website: http://www.wiley.com/go/actuarial

One of our main goals in this chapter is to provide methods for analyzing transactions in terms of their cash flow vectors. There are several basic questions that could be asked:

- When is a transaction worthwhile undertaking?
- How much should one pay in order to receive a certain sequence of cash flows?
- How much should one charge in order to provide a certain sequence of cash flows?
- How does one compare two transactions to decide which one is preferable?

All of these questions are related, and we could answer all of them if we could find a method to put a value on a sequence of future cash flows. If all cash flows were paid at the same time, or if the value of money did not depend on the time that a payment was made, the problem would reduce to one of simple addition. We could simply value a cash flow sequence by adding up all the payments. We cannot proceed in this naive way, however and must consider the *time value of money*. It is a basic economic fact that we prefer present to future consumption. We want to eat the chocolate bar now, rather than tomorrow. We want to enjoy the new car today, rather than next month. This means of course that money paid to us today is worth more than money paid in the future. We are no doubt all very familiar with this fact. We pay interest for the privilege of borrowing money today, which lets us consume now, or we advance money to others, giving up our present consumption and expecting to be compensated with interest earnings. In addition, there is the effect of risk. If we are given a unit of money today, we have it. If we forego it now in return for future payments, there could be a chance that the party who is supposed to make remittance to us may be unable or unwilling to do, and we expect to be compensated for the possible loss. A major step in answering the above questions is to quantify this dependence of value on time.

Readers who have taken courses on the theory of compound interest will be familiar with many of the ideas. However, our treatment will be somewhat different than that usually given. One reason for this is that we want to develop the concepts in such a way that they are applicable to more general situations, as given in Chapters 3–5. A second reason is that our approach is designed to be compatible with modern-day computing methods such as spreadsheets.

To conclude this section, we remark that many complications arise when the cash flows are not exactly known in advance. They may depend on several factors, including random elements. There may be complicated interrelationships between the various cash flows. These matters involve advanced topics in finance and actuarial mathematics and for the most part will not be dealt with in this book. In Part I we deal mainly with a simplified model, where all cash flows are fixed and known in advance. In later parts of the book we will consider certain aspects of randomness, but will not get into the full extent of complications that can arise.

2.2 An analogy with currencies

To motivate the basic ideas, we will consider first a completely different problem, which is nonetheless related to that introduced above. Suppose that I give you 300 Canadian dollars, 200 US dollars and 100 Australian dollars. How much money did I give you? It would be naive indeed to claim that you received 600 dollars, for clearly the currencies are of different value. To answer the question we will need conversion factors that allow us to deduce the value of each type of dollar in terms of others. Let $v(c, u)$ denote the value in Canadian dollars of one US dollar. Assume that $v(c, u) = 1.05$, which means that a US dollar is worth 1.05 Canadian dollars. (Our numbers here are for purposes of illustration only. They are close to the conversion rates at the time of writing, but they may well have changed considerably by the time you are reading this.) Similarly, letting a stand for Australian, we will assume that $v(c, a)$ equals 0.95, which means 95 cents Canadian will buy one Australian dollar. The convention we are using here, which should be noted for later use, is that the v function returns the value of *one* unit of the *second* coordinate currency in terms of the *first* coordinate currency.

There are four more conversion factors of interest, but an important fact is that they can all be deduced from just these two (or indeed from any two that have a common first or common second coordinate). We note first that if it takes 1.05 Canadian dollars to buy 1 US dollar, then a single Canadian dollar is worth $1/1.05 = 0.9524$ US dollars. That is,

$$v(u, c) = v(c, u)^{-1} = 0.9524, \qquad v(a, c) = v(c, a)^{-1} = 1.0526,$$

where we use similar reasoning for the Australian dollar.

Next consider $v(u, a)$. We want the amount of US dollars needed to buy one Australian dollar. We could conceivably effect this purchase in two stages, first using US money to buy Canadian, and then using Canadian to buy Australian. Working backwards, it will take 0.95 Canadian to buy 1 Australian, and it will take $v(u, c)$ 0.95 US dollars to buy the 0.95 Canadian. To summarize,

$$v(u, a) = v(u, c)v(c, a) = 0.9048.$$

Our calculations are completed with

$$v(a, u) = v(u, a)^{-1} = 1.1052.$$

The reader may notice, given a typical real-life listing of currency prices, that the relationships we state here do not hold exactly, but that is due to commissions and other charges. In the absence of these, they must necessarily hold.

Let us now return to the original problem of determining of how much I paid you. We must first select a currency to express the answer in. For example, we could say that the total was equivalent to $300 + 200v(c, u) + 100v(c, a) = 605$ Canadian dollars. We could also say that the total was equivalent to $300v(u, c) + 200 + 100(u, a) = 576.20$ US dollars. Notice as a shortcut, that we did not need to compute the latter sum (which could be a significant saving in calculation if we had several rather than just three currencies). If the total amount is equivalent to 605 Canadian, then it must also be equivalent to $605v(u, c) = 576.20$ US dollars. Similarly, the total in Australian dollars can be computed as $605v(a, c)$ or alternatively as $576.20v(a, u)$, both of which are equal to 637 (approximately as there are some rounding differences).

2.3 Discount functions

We now go back to the original situation. We want to value a sequence of cash flows, which are all in the same currency, but which are paid at different times. Conversion factors are needed to convert the value of money paid at one time to that paid at another. The principles involved